Pivotality, twisted centres, and the anti-double of a Hopf monad

A tale of string diagrams, categories, and monads.

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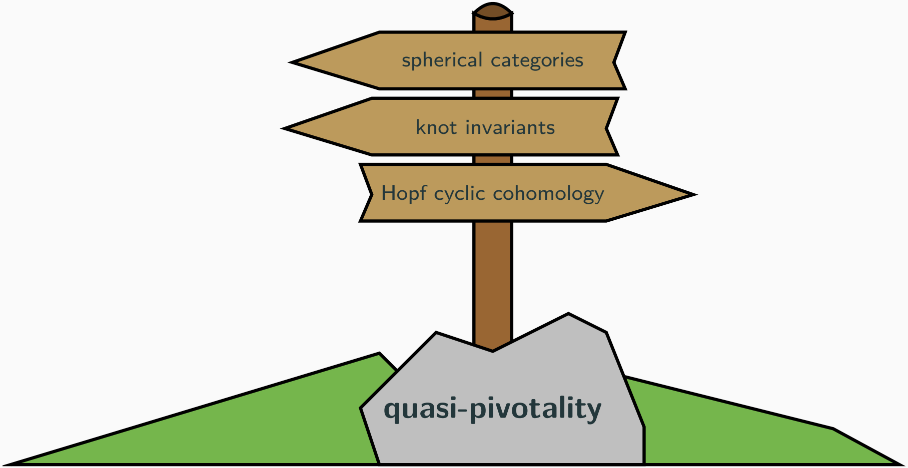
Motivation


spherical categories
knot invariants
Hopf cyclic cohomology
quasi-pivotality

Categories

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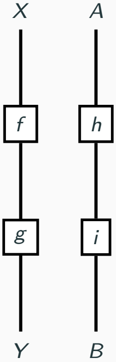
Examples: Set, Vect_k, $[\mathcal{D}, \mathcal{D}]$.

Monoidal categories

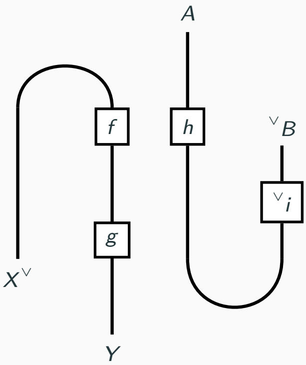
We fix a category $\mathcal{C}$ and equip it with the *monoidal structure* $(\otimes, 1)$ ...



Examples: $(\mathsf{Set}, \times, \{*\})$, $(\mathsf{Vect}_k, \otimes_k, k)$, $([\mathcal{D}, \mathcal{D}], \circ, \mathrm{Id})$.

Rigid categories

We fix a category $\mathcal{C}$ and equip it with the monoidal structure $(\otimes, 1)$, such that *duals* exist.



Examples: $\mathbb{1} \leq (\mathsf{Set}, \times, \{*\})$, $\mathsf{vect}_k \leq (\mathsf{Vect}_k, \otimes_k, k)$, $\mathsf{Ad}^\infty(\mathcal{C}) \leq ([\mathcal{D}, \mathcal{D}], \circ, \mathrm{Id})$.

Module categories

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Bimodule categories

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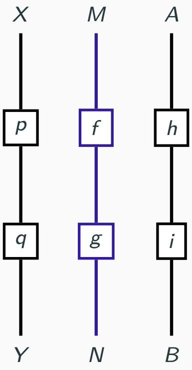
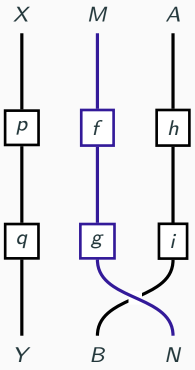


Braided bimodule categories

We consider a second category $\mathcal{M}$ over $\mathcal{C}$ with a left and right action and pass to the *centre* $\mathrm{Z}(\mathcal{M})$.



The motivating example

Examples for $\mathcal{M}$ and $Z(\mathcal{M})$ include

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- For a strict monoidal endofunctor $T \colon \mathcal{C} \longrightarrow \mathcal{C}$,

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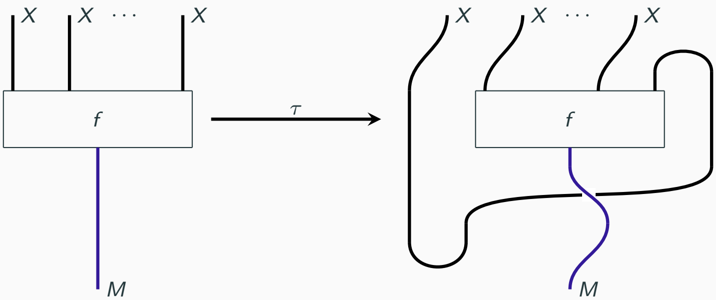
  For T the bidualising functor, $\mathrm{Q}(\mathcal{C}) := \mathrm{Z}(\mathcal{M})$ is the *anti-Drinfeld centre*.
- The Drinfeld centre $\mathrm{Z}(\mathcal{C})$ is braided monoidal and acts on the anti-Drinfeld centre $\mathrm{Q}(\mathcal{C})$.

A cyclic action



The anti-Drinfeld centre induces cyclic permutations.

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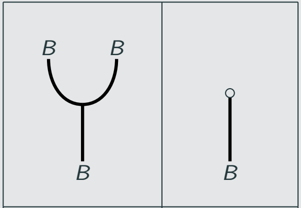

$\mu\colon B^2 \longrightarrow B$	$\eta\colon \mathrm{Id} \longrightarrow B$

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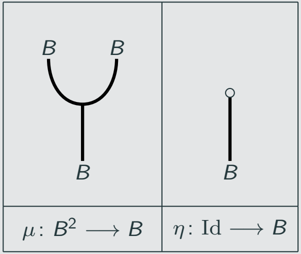
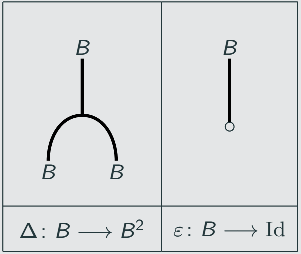
A *bimonad* $B\colon \mathcal{V} \longrightarrow \mathcal{V}$ consists of a monad (B, μ, η) and, morally, a compatible comonoid structure (B, Δ, ε).





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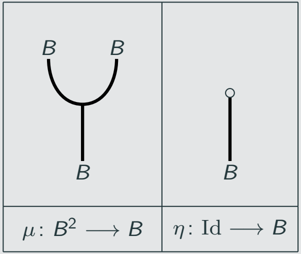
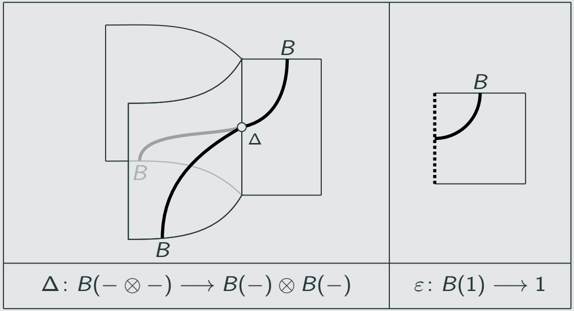
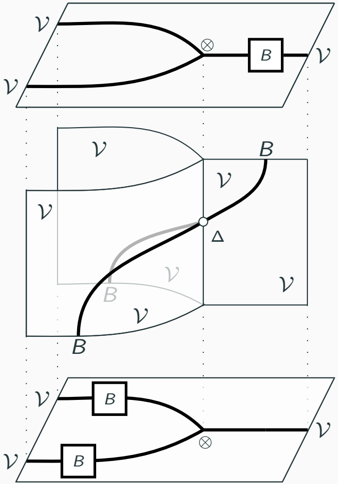
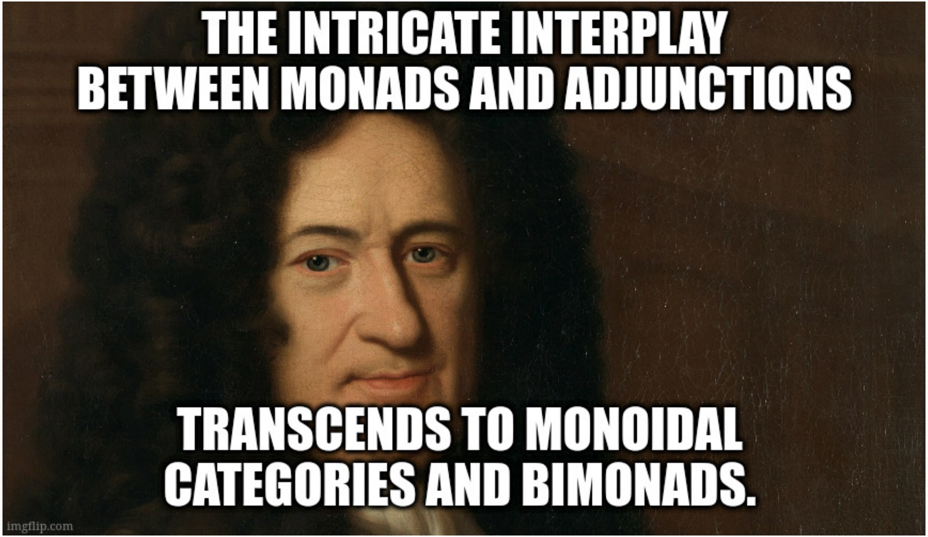




Diagrammatic bimonads: a closer look



Bimonads and adjunctions



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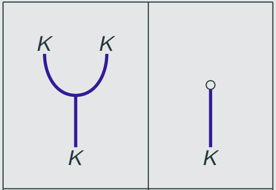
A *comodule monad* $K\colon \mathcal{V} \longrightarrow \mathcal{V}$



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Comodule monads

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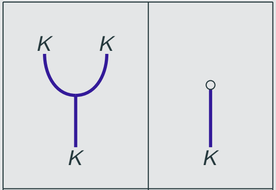
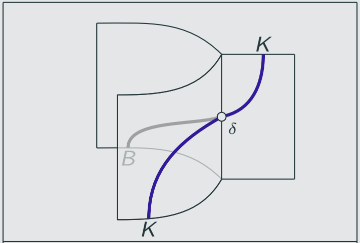
A *comodule monad* $K\colon \mathcal{V} \longrightarrow \mathcal{V}$ over a bimonad $B\colon \mathcal{V} \longrightarrow \mathcal{V}$ consists of:

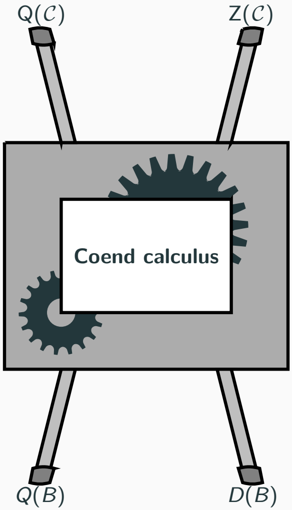


$\mu\colon K^2 \longrightarrow K$ $\quad$ $\eta\colon \mathrm{Id} \longrightarrow K$



$\delta\colon K(- \otimes -) \longrightarrow K(-) \otimes B(-)$

Coend machinery



The central and anti-central monad

Theorem ([BV12, Section 6.2], [HZ22, Theorem 6.16])

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as monoidal and module categories, respectively.

This was just an appetiser—the main course is to be found in our paper.

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- If there are 'trivial' objects in the anti-Drinfeld centre. The biduality functor of the Drinfeld centre is particularly well-behaved [HZ22, Thm. 4.22].
- The existence of these objects can be detected by the anti-Drinfeld double of the identity Hopf monad [HZ22, Cor. 6.27].

Thanks!

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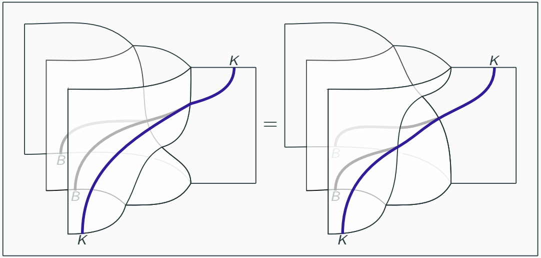

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