Visual Category Theory

On Comodule Monads and Twisted Centers

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A Theorem

Theorem ([Hal21, Theorem 3.4])

For a finite dimensional Hopf algebra, the following are equivalent

1. *The Hopf algebra admits a pair in involution.*
2. *There exists a one dimensional anti-Yetter–Drinfel'd module.*
3. *We have an isomorphism of algebras between the Drinfel'd center and the anti-Drinfel'd center.*

Basic Category Theory

Definition

A *category* $\mathcal{C}$ is ...

Arrows Between Categories

Definition

A *functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is ...

Arrows Between Arrows

Definition

A *natural transformation* $\eta \colon F \Longrightarrow G$ ...

“Nice” Pairs of Functors

Definition

An *adjunction* $F: \mathcal{C} \rightleftarrows \mathcal{D} : U$ consists of the two natural transformations

- $\eta\colon \mathrm{Id}_{\mathcal{C}} \Longrightarrow UF$,
- $\varepsilon\colon FU \Longrightarrow \mathrm{Id}_{\mathcal{D}}$,

that fulfill the *snake identities*.

Monads

Definition

A *monad* (T, μ, η) consists of

- an endofunctor $T\colon \mathcal{C} \longrightarrow \mathcal{C}$
- an associative and unital multiplication $\mu\colon T^2 \Longrightarrow T$,
- a unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \Longrightarrow T$.

Monads as Coordinate Systems

Monads $\leftrightsquigarrow$ Categories

Bimonads $\leftrightsquigarrow$ Monoidal Categories

Hopf Monads $\leftrightsquigarrow$ Rigid Monoidal Categories

Comodule Monads $\leftrightsquigarrow$ Module Categories

A Third Direction of Composition

Definition

A *monoidal category* $(\mathcal{C}, \otimes, 1)$ consists of

- a category $\mathcal{C}$,
- an associative and unital multiplication $\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,
- a unit object 1.

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Translating Categorical Concepts

Definition

A *comonoidal functor* is a triple (F, F_2, F_0) comprising

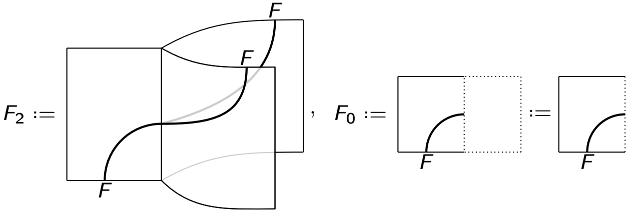
- A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$.
- An associative natural transformation

$$F_{2,X,Y}\colon F(X \otimes Y) \longrightarrow FX \otimes FY, \text{ for all } X, Y \in \mathcal{C},$$

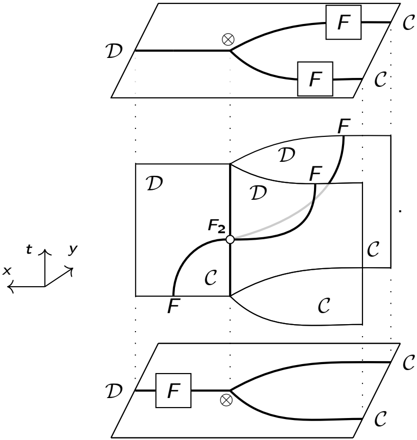
- A morphism $F_0\colon F1 \longrightarrow 1$.

A comonoidal functor is called *strong* (*strict*) iff both additional arrows are isomorphisms (identities).

A Definition Without Words



A closer look at F_2



Naturality

Definition

A *comonoidal natural transformation* between the comonoidal functor F and G consists of

- a natural transformation $\eta \colon F \Longrightarrow G$

that commutes with the monoidal structure of both functors.

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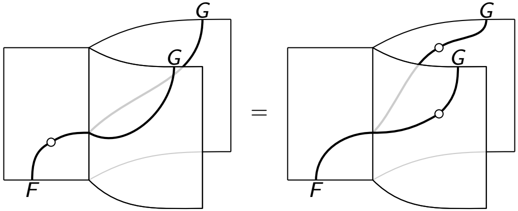

Bimonads

Definition

A *bimonad* on the monoidal category $\mathcal{C}$ consists of

- a monad (B, μ, η) on $\mathcal{C}$,
- a comonoidal functor (B, B_2, B_0) on $\mathcal{C}$,

such that μ and η are comonoidal natural transformations.

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Theorem ([Moe02])

Given a monad T on a monoidal category $\mathcal{C}$, there is a bijective correspondence

$$\{\textit{Bimonad structures on } T\} \xleftrightarrow{1-1} \{\textit{strict monoidal functors } U_T\}$$

Two Useful Observations

1. Categories can act on other categories.

$$\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \quad \rightsquigarrow \quad \triangleright \colon \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$$

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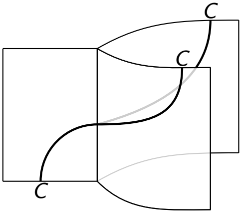
2. Colours are fun.

Purple Functors

$C_2 :=$ 

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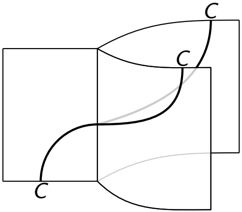
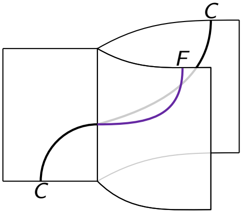
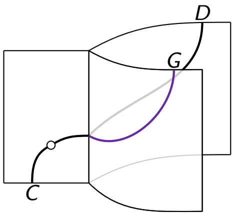


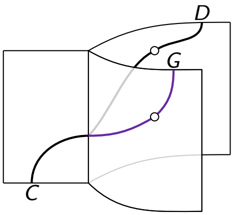
$\lambda :=$



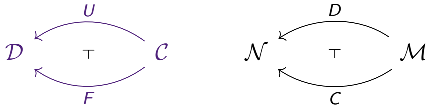
Comodule Natural Transformations



$=$



Comodule Adjunctions



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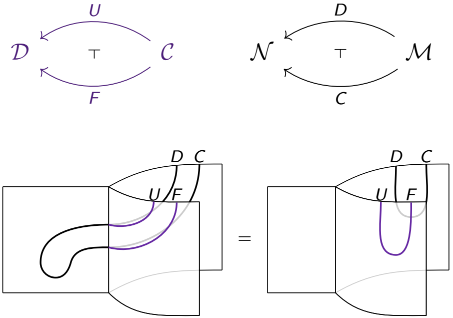

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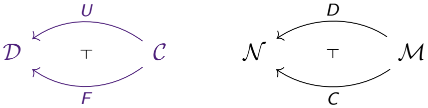
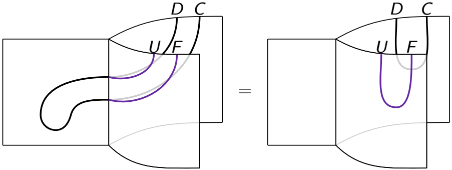
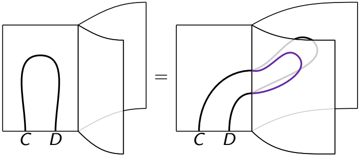


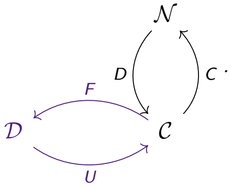




A Classification Result

Theorem

Suppose we have



There is a bijective correspondence

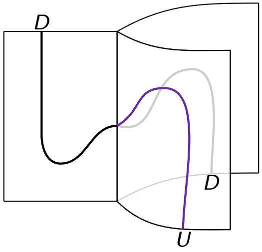
(i) *Lifts of $C \dashv D$ to a comodule adjunction,*

(ii) *Lifts of D to a strong comodule functor from $\mathcal{N}$ to $\mathcal{C}$.*

A Taste of Graphical Proofs

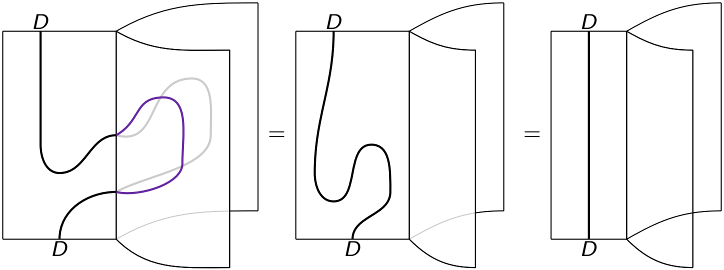
(I) $\implies$ (II): Assume $C \dashv D$ is a comodule adjunction.

Define $(\lambda^D)^{-1}$ as



A Taste of Graphical Proofs^2

The natural transformation $(\lambda^D)^{-1}$ is a post-inverse of λ^D:



Comodule Monads

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Let (B, μ, η) be a bimonad on $\mathcal{C}$. A *comodule monad over B* is a comodule endofunctor (C, λ) on $\mathcal{M}$ over B such that

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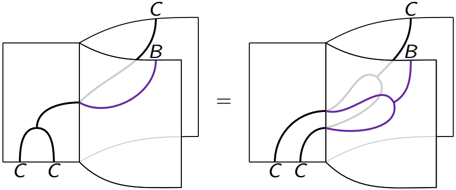

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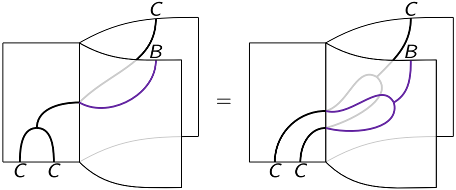

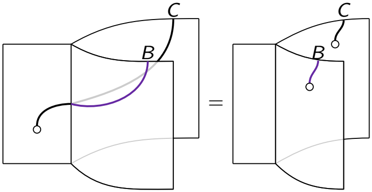

Reconstruction For Comodule Monads

Theorem

Let B and C be a bimonad respectively monad on $\mathcal{C}$. There is a bijective correspondence between

(i) *Comodule structure on C,*

(ii) *Left modules structures of $\mathcal{C}^C$ over $\mathcal{C}^B$ such that $U_C(\triangleright) = \otimes$.*

Lifting Theorems

Theorem ([BV07, Theorem 5.6])

Assuming a nice enough base category $\mathcal{C}$, the center of the category $\mathcal{C}$ gives rise to a Hopf monad

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Assuming a nice enough base category $\mathcal{C}$, the center of the category $\mathcal{C}$ gives rise to a Hopf monad and twisted center gives rise to a comodule monad.

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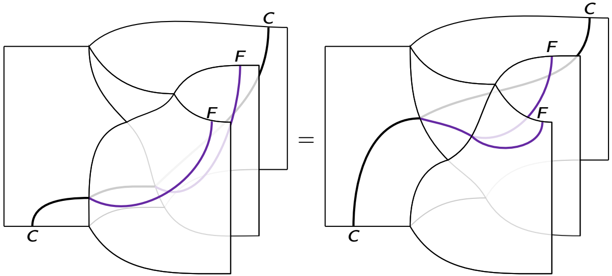
Theorem

For a finite ~~dimensional Hopf algebra~~ tensor category, the following are equivalent

1. *The ~~Hopf algebra~~ category admits a ~~pair in involution~~ quasi-pivotal struture.*
2. *There exists ~~a one dimensional anti-Yetter–Drinfel'd module~~ an "invertible" element in the twisted center.*
3. *We have an ~~isomorphism~~ equivalence of ~~algebras~~ categories between the ~~Drinfel'd~~ center and the ~~anti-Drinfel'd~~ twisted center.*

Bonus: Associativity in Three Dimensions



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[Moe02] Ieke Moerdijk. “Monads on tensor categories”. In: vol. 168. 2-3. Category theory 1999 (Coimbra). 2002, pp. 189–208. doi: 10.1016/S0022-4049(01)00096-2. url: https://doi.org/10.1016/S0022-4049(01)00096-2.